

Muon-proton Scattering

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Abstract

A recent proposal to measure the proton form factor by means of muon-proton scattering will use muons which are not ultrarelativistic (and also not nonrelativistic). The usual equations describing the scattering cross section use the approximation that the scattered lepton (usually an electron) is ultrarelativistic, with v/c approximately equal to 1. Here the cross section is calculated for all values of the energy. It agrees with the standard result in the appropriate limit.

Introduction

A proposal for muon-proton scattering at PSI [1] has been made in an attempt to help resolve the proton radius puzzle. A measurement made on the basis of the Lamb Shift in muonic hydrogen [2] disagrees with the radius measured in atomic spectroscopy and electron scattering experiments [3, 4, 5]. The proposal will directly test whether or not $\mu - p$ and $e - p$ scattering are the same and will perform measurements with μ^\pm and e^\pm at low Q^2 in order to study the two-photon exchange contributions in greater detail. Since the muon is about 206.7 times heavier than the electron [6], for the energies mentioned in the proposal, the muons are neither ultrarelativistic nor nonrelativistic. For the muon momenta given in the proposal the value of v/c for the incoming lepton is between 0.7 and 0.9, while the standard expressions for the scattering cross section are valid only for v/c very close to 1. The standard kinematics assumptions made in the analysis of e-p scattering will not all be valid in the case of an experiment on mu-p scattering. The cross sections have been calculated without these approximations [7, 8], but since these results seem to have been forgotten, it is worth presenting another calculation of the basic cross section without them.

According to the proposal, scattering of negative and positive muons (and electrons) will be studied. The muon momenta will be in the range (115-210) MeV/c with scattering angles in the range 20° to 100° , corresponding to Q^2 in the range (0.01-0.1) (GeV/c)². For comparison, $m_\mu^2 c^2 = 0.01116$ (GeV/c)².

Numerical values for the muon energy ($E = \sqrt{p^2 + m^2}$) and velocity ($\beta = |\vec{p}|/E = v/c$) corresponding to the incoming momenta in the proposal are given by:

p (MeV/c)	E(MeV)	β
115	156.17	0.7364
153	185.94	0.8229
210	235.08	0.8933

Obviously the approximation $\beta \approx 1$ is not valid for the energies considered in the proposal. The radiative corrections to the scattering cross section are functions of Q^2/m_μ^2 , which is in the range of approximately 0.9-9.0. The usual formulas [10, 11, 12], which assume that $Q^2/m^2 \gg 1$, will not be accurate.

The convention of Bjorken and Drell [9] will be used. The metric used is defined by $p_i \cdot p_j = E_i E_j - \vec{p}_i \cdot \vec{p}_j$. Also m is the lepton rest mass, M is the target rest mass, and $\alpha = e^2/4\pi$. Use p_1 and p_3 for the incoming and outgoing muon four-momenta, and p_2 and p_4 for the incoming and outgoing proton four-momenta, respectively. In the lab system we have $p_1 = (E, \vec{p})$, $p_3 = (E', \vec{p}')$, $p_2 = (M, 0)$, $p_4 = (M + \omega, \vec{q})$. Here $q = p_1 - p_3 = p_4 - p_2$, and $\omega = q_0 = E - E'$. It is useful to observe that $q^2 = 2m^2 - 2p_1 \cdot p_3 = 2M^2 - 2p_2 \cdot p_4 = -2M\omega$. This is simply a result of energy conservation. Since q^2 is negative with the metric used here, it is sometimes convenient to define $Q^2 = -q^2$. The proton current is taken to have the usual on-shell form, characterized by

$$\Gamma_\mu = F_1(q^2)\gamma_\mu + \kappa F_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}$$

Here κ is the anomalous magnetic moment of the proton. The so-called Sachs form factors are related to F_1 and F_2 by $G_M = F_1 + \kappa F_2$, $G_E = F_1 - \frac{Q^2}{4M^2}\kappa F_2$.

Or, $F_1 = (G_E + \frac{Q^2}{4M^2}G_M)/(1 + \frac{Q^2}{4M^2})$ and $\kappa F_2 = (G_M - G_E)/(1 + \frac{Q^2}{4M^2})$

For the calculation of the matrix element, the Gordon decomposition ([9])

$$\bar{u}(p_4)\frac{i\sigma_{\mu\nu}q^\nu}{2M}u(p_2) = \bar{u}(p_4)[\gamma_\mu - \frac{(p_2 + p_4)_\mu}{2M}]u(p_2)$$

is very useful. As a result, one may write

$$\bar{u}(p_4)\Gamma_\mu u(p_2) = \bar{u}(p_4)[(F_1 + \kappa F_2)\gamma_\mu - \kappa F_2 \frac{(p_2 + p_4)_\mu}{2M}]u(p_2) \quad (1)$$

Scattering Cross Section

Following Chap. 7 of Ref. [9], the invariant matrix element for scattering of a charged lepton from a proton in Born approximation is given by

$$\mathfrak{M}_{fi} = \bar{u}(p_3)\gamma^\mu u(p_1)\frac{e^2}{q^2}\bar{u}(p_4)\Gamma_\mu u(p_2) \quad (2)$$

The cross section for scattering of the charged lepton into a given solid angle $d\Omega'$ about an angle θ is given by

$$\frac{d\sigma}{d\Omega'} = \frac{m^2 M^2}{4\pi^2} \frac{p'/p}{M + E - pE'/p'\cos\theta} |\mathfrak{M}_{fi}|^2 \quad (3)$$

In Eq. 3 it is assumed that $|\mathfrak{M}_{fi}|^2$ has been averaged over initial spins and summed over final spins. The final result is given by

$$\frac{d\sigma}{d\Omega'} = \frac{\alpha^2}{q^4} \frac{p'/p}{1 + (E - pE'/p' \cos \theta)/M} \left[G_E^2 \frac{(4EE' + q^2)}{1 - q^2/4M^2} + G_M^2 \left((4EE' + q^2) \left(1 - \frac{1}{1 - q^2/4M^2} \right) + \frac{q^4}{2M^2} + \frac{q^2 m^2}{M^2} \right) \right] \quad (4)$$

Recall that generally $-q^2 = Q^2 = 2M(E - E') = 2(EE' - pp' \cos \theta - m^2)$

If the limit of very high lepton energies, one has

$$p \approx E, \quad p' \approx E', \quad q^2 \approx -4EE' \sin^2(\theta/2)$$

In this case, the cross section given in Eq. 4 reduces to

$$\frac{d\sigma}{d\Omega'} = \frac{\alpha^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)} \frac{1}{1 + 2E \sin^2(\theta/2)/M} \left[\frac{Q^2}{2M^2} G_M^2 \left(\frac{1}{1 + Q^2/4M^2} + 2 \tan^2(\theta/2) \right) + \frac{G_E^2}{1 + Q^2/4M^2} \right] \quad (5)$$

This agrees with the expression given in Sec.4 of ref. [13] (and with equivalent expressions in other work).

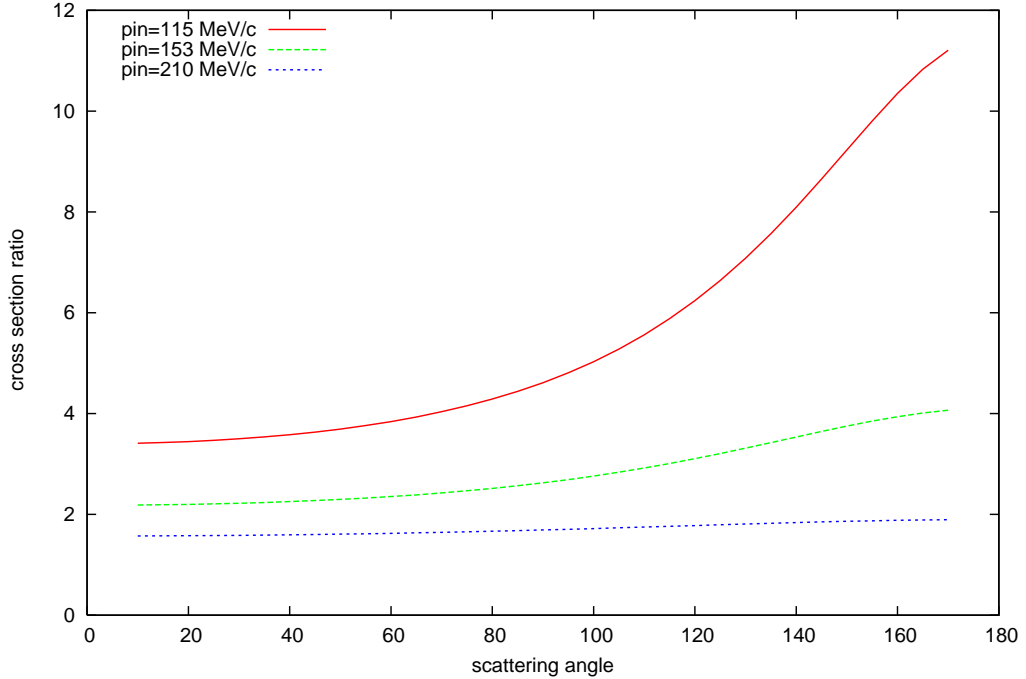


Figure 1: Ratio of the exact scattering cross section to the cross section calculated with the commonly used relativistic approximation.

Figure 1 shows the ratio of the cross section calculated with the exact formula (Eq. 4) to that calculated with Eq. 5, but with exact muon kinematics. The form factors were taken from a parametrization by [14]. The difference is significant, especially at lower values of incident momentum.

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Appendix

Here a few details of the calculation of the average over initial spins and sum over final spins of $|\mathfrak{M}_{fi}|^2$ are given. \mathfrak{M}_{fi} is given by Eq. 2.

Following Chap. 7 of Ref. [9] the spin sum and average of $|\mathfrak{M}_{fi}|^2$ is given by

$$|\mathfrak{M}_{fi}|^2 = \frac{e^4}{4(q^2)^2} \cdot \text{Tr} \left[\left(\frac{\not{p}_3 + m}{2m} \right) \gamma^\mu \left(\frac{\not{p}_1 + m}{2m} \right) \gamma^\nu \right] \cdot \text{Tr} \left[\left(\frac{\not{p}_4 + M}{2M} \right) \Gamma_\mu \left(\frac{\not{p}_2 + M}{2M} \right) \Gamma_\nu \right] \quad (6)$$

The lepton trace is

$$\frac{1}{m^2} [p_3^\mu p_1^\nu + p_1^\mu p_3^\nu + g^{\mu\nu} q^2/2]$$

The trace for the proton is

$$G_M^2 \text{Tr} \left[\left(\frac{\not{p}_4 + M}{2M} \right) \gamma_\mu \left(\frac{\not{p}_2 + M}{2M} \right) \gamma_\nu \right] + \left(\frac{\kappa F_2}{2M} \right)^2 (p_2 + p_4)_\mu (p_2 + p_4)_\nu \text{Tr} \left(\frac{\not{p}_4 + M}{2M} \right) \left(\frac{\not{p}_2 + M}{2M} \right) - \frac{G_M \kappa F_2}{2M} \left[(p_2 + p_4)_\mu \text{Tr} \left(\frac{\not{p}_4 + M}{2M} \right) \left(\frac{\not{p}_2 + M}{2M} \right) \gamma_\nu + (p_2 + p_4)_\nu \text{Tr} \left(\frac{\not{p}_4 + M}{2M} \right) \gamma_\mu \left(\frac{\not{p}_2 + M}{2M} \right) \right] \quad (7)$$

where Eq. 1 has been used.

Evaluation of the traces gives

$$\frac{G_M^2}{M^2} [p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} + g^{\mu\nu} q^2/2] + \left(\frac{\kappa F_2}{2M} \right)^2 (p_2 + p_4)_\mu (p_2 + p_4)_\nu \frac{1}{M^2} (p_2 \cdot p_4 + M^2) - \frac{G_M \kappa F_2}{M^2} (p_2 + p_4)_\mu (p_2 + p_4)_\nu \quad (8)$$

The terms involving $\kappa F_2 = (G_M - G_E)/(1 - \frac{q^2}{4M^2})$ are

$$\frac{1}{M^2} (p_2 + p_4)_\mu (p_2 + p_4)_\nu \left[-G_M \kappa F_2 + \left(\frac{\kappa F_2}{2M} \right)^2 (1 - q^2/4M^2) \right]$$

In terms of the Sachs form factors

$$\left(\frac{\kappa F_2}{2M} \right)^2 (1 - q^2/4M^2) - G_M \kappa F_2 = \frac{G_E^2 - G_M^2}{2(1 - q^2/4M^2)}$$

The square of the matrix element becomes

$$\begin{aligned} |\mathfrak{M}_{fi}|^2 &= \frac{4\pi^2 \alpha^2}{m^2 M^2 (q^2)^2} \left[G_M^2 (2p_1 \cdot p_2 p_3 \cdot p_4 + 2p_1 \cdot p_4 p_3 \cdot p_2 + (m^2 + M^2) q^2/2) \right. \\ &\quad \left. + \frac{G_E^2 - G_M^2}{2(1 - q^2/4M^2)} (2p_1 \cdot (p_2 + p_4) p_3 \cdot (p_2 + p_4) + q^2 (p_2 + p_4) \cdot (p_2 + p_4)/2) \right] \\ &= \frac{4\pi^2 \alpha^2}{m^2 M^2 (q^2)^2} \left[G_M^2 (4M^2 E E' + (M^2 + m^2) q^2 + q^4/2) + \frac{G_E^2 - G_M^2}{1 - q^2/4M^2} (4M^2 E E' + M^2 q^2) \right] \quad (9) \end{aligned}$$